

Dynamical behavior of SIR epidemic model with non-integer time fractional derivatives: A mathematical analysis



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ARTICLE INFO

Article history:

Received 31 July 2017

Received in revised form

30 October 2017

Accepted 25 November 2017

Keywords:

Epidemic model

Fractional derivatives

Endemic equilibrium

LADM

Convergence analysis

ABSTRACT

Protection of children from vaccine preventable diseases, such as measles is among primary goal for health worker. Measles is a highly contagious disease that can spread in a population depending on the number of peoples susceptible or infected and also depending on their dynamics in the community. The model monitors the temporal dynamics of a childhood disease in the presence of preventive vaccine. We presented a nonlinear time fractional model of measles in order to understand the outbreaks of this epidemic disease. The Caputo fractional derivative operator of order $\alpha \in (0,1]$ is employed to obtain the system of fractional differential equations. The numerical solution of the time fractional model has been procured by employing Laplace Adomian decomposition method (LADM), qualitative and sensitivity analysis of the model was performed. Qualitative results shows that the model has endemic equilibrium which locally asymptotically stable for $R_0 > 1$ and otherwise unstable. The convergence analysis and non-negative solutions are verified for the proposed scheme. Simulation of different epidemiological classes at the effect of fractional parameter α revealed that most individuals undergoing treatment join the recovered class. This method proves to be very efficient techniques for solving epidemic model to control infectious disease.

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1. Introduction

Epidemiological study plays an important role to understand the impact of infectious disease in a community. In mathematical modeling, we investigate models by model building, perform estimation of parameters, check sensitivity of models by varying parameters and compute their numerical simulations (Panum, 1988). The research of this kind helps to understand the ratio of disease spread in the population and to control their parameters (Grenfell, 1992; Abubakar et al., 2012). These types of diseased models are often called infectious diseases (i.e., the disease which transferred from one person to another person). Measles, rubella, chicken pox, mumps, aids and gonorrhoea syphilis are the examples of infectious disease (WHO, 2011). Rubeola virus is highly infectious illness which is also known as morbilli or measles. The virus can be

found in the mucus of the throat, nose of an infected adult and child. Measles symptoms caused by Rubeola virus always included fever, coryza (runny nose), conjunctivitis and at least one of the three Cough. Symptoms appear after the initial infection about 9-11 days (Ochoche and Gweryina, 2014). Complications of measles are fairly common but the patients have weak immune system are more likely to be worse such as those with HIV/AIDS or leukemia and those with vitamin deficiency. Healthy children over the age of 5 are less likely to have complications than adults over the age of 20. It is the first and worst eruptive fever occurs during childhood (Hethcote, 2000; Murray, 2002). It produces eye infection, bronchitis, laryngitis and vomiting, bronchitis is inflammation of the inner walls of airways and laryngitis is inflammation of the voice box.

Since in recent years fractional calculus has attracted great attentions from researchers and different aspects of the said subject is under consideration for research. This is due to the fact that fractional derivative is important tool to explain the dynamical behavior of various physical systems. The strength of this differential operator is their

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<https://doi.org/10.21833/ijaas.2018.01.016>

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nonlocal characteristics which do not exist in the integer order differential operators. The distinguished features of fractional differential equations are that it outlines memory and transmitted properties of numerous mathematical models. As a fact, that fractional order models are more realistic and practical than the classical integer order models. Fractional order derivative produces greater degree of freedom in these models. Arbitrary order derivatives are powerful tools for the discretion of the dynamical behavior of various biomaterial and systems. The most iterating feature of these models is their global (nonlocal) characteristics which do not exist in the classical order models (Haq et al., 2017).

Laplace transform method is a useful technique in different field of biological science, engineering and applied mathematics. The coupling of ADM and Laplace transform leads to a powerful method known as Laplace Adomain decomposition method. With the help of Laplace transform, we convert a differential equation to an algebraic equation and the nonlinear terms are decomposed in terms of Adomain polynomials. The given numerical technique works powerfully for a system of deterministic as well as stochastic differential equations. More unambiguously, it can be used for classical as well as fractional order system of linear and nonlinear ordinary and partial differential equations. In this method, no perturbation or liberalization is required. Further it has no need of pre-defined step size like RK4. Also, this method does not depend upon on a parameter like needed for homotopy perturbation method (HPM) and homotopy analysis method (HAM). Although the solutions obtained via this method are the same as obtained by ADM, for detail see (Jafari et al., 2011a; 2011b; Johnston et al., 2016). It is to be noted that LADM is powerful than standard ADM method (Haq et al., 2017).

2. Material and method

2.1. Mathematical model

Kermack and McKendrick (1927) are inventor of the diseases models and played an important role in mathematical epidemiology to describe the transmission dynamics of measles formulate a deterministic and compartmental mathematical model. The population is consistently fraternization and replicates the demography of a typical emerging country, as it investigates an exponentially increasing dynamics. The total population (N) is divided into three class's s, i and r represented as susceptible, infected and recovered population respectively to describe in equations of the model (Akinboro et al., 2014).

$$\begin{aligned} \frac{ds}{dt} &= \mu N - \beta s(t)i(t) - \mu s(t) & (1) \\ \frac{di}{dt} &= \beta s(t)i(t) - (\gamma + \mu)i(t) & (2) \\ \frac{dr}{dt} &= \gamma i(t) - \mu r(t) & (3) \end{aligned}$$

where μ is per capita removal rate, β is transitivity, γ is per capita recovery rate. Since μ , β and γ are interpreted rates, their ranges are $0 \leq \mu \leq 1$, $0 \leq \beta \leq 1$ and $0 \leq \gamma \leq 1$ (Podlubny, 1999).

The fractional order extension of this model has been first studied in (Podlubny, 1999; Arqub and El-Ajou, 2013). The aim of using fractional system of differential equations (FDEs) is naturally related to systems with memory effects which exist in most biological system that shows the realistic biphasic decline behavior of infection or diseases but at a slower rate. The new system is fractional differential equations (FDEs) which are described as follows.

$$D^{\alpha_1} s(t) = \mu N - \beta s(t)i(t) - \mu s(t) \tag{4}$$

$$D^{\alpha_2} i(t) = \beta s(t)i(t) - (\gamma + \mu)i(t) \tag{5}$$

$$D^{\alpha_3} r(t) = \gamma i(t) - \mu r(t) \tag{6}$$

subject to initial conditions

$$s(0) = n_1, i(0) = n_2, r(0) = n_3 \tag{7}$$

the initial conditions must satisfied $N = s + i + r$ or $N = n_1 + n_2 + n_3$.

2.2. Preliminaries

In this section, we give some fundamental results and definitions from fractional calculus. For detailed over view of the topic readers are referred to (Haq et al., 2017; Jafari et al., 2011a; 2011b; Johnston et al., 2016).

Definition 1: The Riemann-liouville fractional integration of order α is defined as

$$(J_{t_0}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{\alpha} (t-s)^{\alpha-1} f(s) ds, \alpha > 0, t > t_0$$

$$(J_{t_0}^{\alpha} f)(t) = f(t)$$

The Riemann-Liouville derivative has certain disadvantages such that the fractional derivative of a constant is not zero. Therefore, we will make use of Caputo's definition owing to its convenience for initial conditions of the fractional differential equations.

Definition 2: The Riemann-liouville fractional integration of order α is defined as

$$D^{\alpha} f(t) = D^n (J^{n-\alpha} f(t)),$$

$$D_*^{\alpha} f(t) = J^{n-\alpha} (D^n f(t))$$

where $n - 1 < \alpha \leq n$, $n \in N$, f is the given function, It is known that $(J_{t_0}^{\alpha} f)(t) \rightarrow f(t)$ as $\alpha \rightarrow 1$.

Definition 3: The definitions of Laplace transform of Caputo's derivative and Mittag-Leffler function in two arguments is written as

$$L\{D^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{i=0}^{n-1} s^{\alpha-i-1} f^i(0), n - 1 < \alpha \leq n, n \in N$$

3. Mathematical analysis

The system is qualitatively analyzed by two ways i.e. disease Free Equilibrium and endemic Equilibrium. To evaluate the equilibrium point, we take

$$D^{\alpha_1} s(t) = D^{\alpha_2} i(t) = D^{\alpha_3} r(t) = 0. \tag{8}$$

Therefore the disease free equilibrium is

$$P_0 = (s, i, r) \text{ i.e. } (s, i, r) = (N, 0, 0)$$

The Equilibrium which is not disease free i.e., Endemic Equilibrium (EE). This state that the disease persists in a population and never dies out. If

$$i \neq 0$$

then,

$$s = \frac{\gamma + \mu}{\beta}, r = \frac{\beta\gamma N - \gamma(\gamma + \mu)}{\beta\gamma(\gamma + \mu)}, r = \frac{\beta\mu N - \mu(\gamma + \mu)}{\beta(\gamma + \mu)}.$$

Thus the endemic equilibrium state is given as:

$$(s, i, r) = \left(\frac{\gamma + \mu}{\beta}, \frac{\beta\gamma N - \gamma(\gamma + \mu)}{\beta\gamma(\gamma + \mu)}, \frac{\beta\mu N - \mu(\gamma + \mu)}{\beta(\gamma + \mu)} \right)$$

Theorem 3.1: The disease-free equilibrium P_0 is locally asymptotically stable if $R_0 < 1$ and is unstable if $R_0 > 1$

Theorem 3.2: The endemic equilibrium state $E_1 = (s^*, i^*, r^*)$ of the model (4)-(5) is locally asymptotically stable if $R_0 > 1$, otherwise unstable.

3.1. Reproductive number

In this system the threshold result of this equilibrium is $R_0 > 1$, so this is in endemic state. Consider the Jacobean matrix as

$$J = \begin{bmatrix} -(\beta i + \mu) & -\beta s & 0 \\ \beta i & \beta s - (\gamma + \mu) & 0 \\ 0 & \gamma & -\mu \end{bmatrix}$$

$$J = F - V$$

were

$$F = \begin{bmatrix} -\beta i & -\beta s & 0 \\ \beta i & \beta s & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$V = \begin{bmatrix} \mu & 0 & 0 \\ 0 & (\gamma + \mu) & 0 \\ 0 & -\gamma & \mu \end{bmatrix}$$

By using the relation $|K - \lambda I| = 0$ where $K = FV^{-1}$ and got the eigen value $\lambda = \frac{\beta N}{\gamma + \mu}$ which represents the reproductive number $R_0 = \frac{\beta N}{\gamma + \mu}$

3.2. Non negative solution

Let $R_+^3 = \{x \in R^3, x \geq 0\}$ and $x(t) = (s(t), i(t), r(t))^T$ For its proof, we need to use the followings lemma.

Lemma: Let $h(x) \in C[a, b]$ and $D^\alpha h(x) \in C[a, b]$ for $0 < \alpha < 1$ then, we have

$$h(s) = h(a) + \frac{1}{(a+1)!} D^\alpha h(\eta)(x - a) \tag{9}$$

with $0 \leq \eta \leq x$ for all $x \in (a, b]$

Theorem 3.3: There is a unique solution for the initial value problem given by (4)-(6), and the solution remains in $R^3, x \geq 0$.

Proof: The uniqueness and existence for the solution of (4)-(6), in $(0, \alpha)$ can be obtained. Our aim is to show the domain $R^3, x \geq 0$ is positively invariant. Since

$$D^\alpha s|_{s=0} = \mu N \geq 0 \tag{10}$$

$$D^\alpha i|_{i=0} = 0 \tag{11}$$

$$D^\alpha r|_{r=0} = \gamma i \geq 0. \tag{12}$$

The nonnegative solution satisfied the vector field points into R_+^3 .

Theorem 3.4: E_0 is locally asymptotically stable if $Re(\lambda) < 0$.

Proof: For the system (4)-(6) the $|J - \lambda I| = 0$, then we get.

$$\begin{vmatrix} -(\beta i + \mu + \lambda) & -\beta s & 0 \\ \beta i & \beta s - (\gamma + \mu + \lambda) & 0 \\ 0 & \gamma & -\mu - \lambda \end{vmatrix} = 0 \tag{13}$$

$$(\beta i + \mu + \lambda)[\beta s - (\gamma + \mu + \lambda)](\mu + \lambda) + \beta^2 s i (\mu + \lambda) = 0.$$

The equation which is given above is called characteristic equation. But recall that EE is given as:

$$(s, i, r) = \left(\frac{\gamma + \mu}{\beta}, \frac{\beta\gamma N - \gamma(\gamma + \mu)}{\beta\gamma(\gamma + \mu)}, \frac{\beta\mu N - \mu(\gamma + \mu)}{\beta(\gamma + \mu)} \right)$$

$$(\mu + \lambda)[(\beta i + \mu + \lambda)(\beta s - (\gamma + \mu + \lambda)) + \beta^2 s i] = 0.$$

Either

$$(\mu + \lambda) = 0 \text{ or } [(\beta i + \mu + \lambda)(\beta s - (\gamma + \mu + \lambda)) + \beta^2 s i] = 0$$

$$\lambda_2 = -\frac{\mu}{2} + \frac{\sqrt{\mu(4\gamma + 9\mu - 4\beta N)}}{2}, \lambda_2 = -\frac{\mu}{2} - \frac{\sqrt{\mu(4\gamma + 9\mu - 4\beta N)}}{2}$$

$\lambda_i < 0$ or if λ_i is complex with negative real parts then system is stable.

Therefore $Re(\lambda) < 0$, since all the parameters are non-negative. So E_1 is locally asymptotically stable. This proves the proposition.

3.3. Sensitivity analysis of R_0

The sensitivity of R_0 is as follows

$$\begin{aligned} \frac{\partial R_0}{\partial N} &= \frac{\beta}{\gamma + \mu} > 0 \\ \frac{\partial R_0}{\partial \beta} &= \frac{N}{\gamma + \mu} > 0 \\ \frac{\partial R_0}{\partial \mu} &= -\frac{\beta N}{(\gamma + \mu)^2} < 0 \\ \frac{\partial R_0}{\partial \gamma} &= -\frac{\beta N}{(\gamma + \mu)^2} < 0 \end{aligned}$$

It can be seen that R_0 is most sensitive to change in parameter, here, R_0 is increasing with N, β and decreasing with μ, γ .

3.4. The Laplace-Adomian decomposition method

Consider the fractional-order epidemic model (4), (5) and (6) subject to the initial condition (7). The nonlinear term in this model is si and μ, β, γ are known constants. For $\alpha_1 + \alpha_2 + \alpha_3 = 1$ the fractional order model converts to the classical epidemic model. Applying the Laplace transform on both sides of (4), (5) and (6), we get

$$\begin{aligned} L\{D^{\alpha_1} s(t)\} &= \mu NL\{1\} - \beta L\{s(t)i(t)\} - \mu L\{s(t)\} & (14) \\ L\{D^{\alpha_2} i(t)\} &= \beta L\{s(t)i(t)\} - (\gamma + \mu)L\{i(t)\} & (15) \\ L\{D^{\alpha_3} r(t)\} &= \gamma L\{i(t)\} - \mu L\{r(t)\} & (16) \end{aligned}$$

by using the property of Laplace transform, we have

$$\begin{aligned} S^{\alpha_1} L\{s\} - S^{\alpha_1-1} s(0) &= \mu NL\{1\} - \beta L\{s(t)i(t)\} - \mu L\{s(t)\} & (17) \\ S^{\alpha_2} L\{i\} - S^{\alpha_2-1} i(0) &= \beta L\{s(t)i(t)\} - (\gamma + \mu)L\{i(t)\} & (18) \\ S^{\alpha_3} L\{r\} - S^{\alpha_3-1} r(0) &= \gamma L\{i(t)\} - \mu L\{r(t)\} & (19) \\ S^{\alpha_1} L\{s\} &= S^{\alpha_1-1} s(0) + \mu NL\{1\} - \beta L\{s(t)i(t)\} - \mu L\{s(t)\} & (20) \\ S^{\alpha_2} L\{i\} &= S^{\alpha_2-1} i(0) + \beta L\{s(t)i(t)\} - (\gamma + \mu)L\{i(t)\} & (21) \\ S^{\alpha_3} L\{r\} &= S^{\alpha_3-1} r(0) + \gamma L\{i(t)\} - \mu L\{r(t)\} & (22) \end{aligned}$$

by using the initial conditions (7), we get

$$\begin{aligned} L\{s\} &= \frac{n_1}{s} + \frac{\mu N}{s^{\alpha_1+1}} - \frac{\beta}{s^{\alpha_1}} L\{s(t)i(t)\} - \frac{\mu}{s^{\alpha_1}} L\{s(t)\} & (23) \\ L\{i\} &= \frac{n_2}{s} + \frac{\beta}{s^{\alpha_2}} L\{s(t)i(t)\} - \frac{(\gamma + \mu)}{s^{\alpha_2}} L\{i(t)\} & (24) \\ L\{r\} &= \frac{n_3}{s} + \frac{\gamma}{s^{\alpha_3}} L\{i(t)\} - \frac{\mu}{s^{\alpha_3}} L\{r(t)\} & (25) \end{aligned}$$

It should be assumed that method gives the solution as an infinite series

$$s = \sum_{k=0}^{\infty} s_k, \quad i = \sum_{k=0}^{\infty} i_k, \quad r = \sum_{k=0}^{\infty} r_k. \tag{26}$$

The nonlinearity si can be written as

$$si = \sum_{k=0}^{\infty} A_k$$

where A_k is called the Adomian polynomials given as

$$A_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} [\sum_{j=0}^k \lambda^j s_j \sum_{j=0}^k \lambda^j i_j] \tag{27}$$

Substitute equations (26) and (27) in (23)-(25), we have the followings results

$$L\{s_0\} = \frac{n_1}{s} + \frac{\mu N}{s^{\alpha_1+1}}, \quad L\{i_0\} = \frac{n_2}{s}, \quad L\{r_0\} = \frac{n_3}{s} \tag{28}$$

Similarly, we have

$$\begin{aligned} L\{s_1\} &= -\frac{\beta}{s^{\alpha_1}} L\{A_0\} - \frac{\mu}{s^{\alpha_1}} L\{s_0\} \dots, & L\{s_{k+1}\} &= \\ & -\frac{\beta}{s^{\alpha_1}} L\{A_k\} - \frac{\mu}{s^{\alpha_1}} L\{s_k\} & (29) \end{aligned}$$

$$\begin{aligned} L\{i_1\} &= \frac{\beta}{s^{\alpha_2}} L\{A_0\} - \frac{(\gamma + \mu)}{s^{\alpha_2}} L\{i_0\} \dots, & L\{i_{k+1}\} &= \\ & \frac{\beta}{s^{\alpha_2}} L\{A_k\} - \frac{(\gamma + \mu)}{s^{\alpha_2}} L\{i_k\} & (30) \end{aligned}$$

$$\begin{aligned} L\{r_1\} &= \frac{\gamma}{s^{\alpha_3}} L\{i_0\} - \frac{\mu}{s^{\alpha_3}} L\{r_0\} \dots, & L\{r_{k+1}\} &= \\ & \frac{\gamma}{s^{\alpha_3}} L\{i_k\} - \frac{\mu}{s^{\alpha_3}} L\{r_k\} & (31) \end{aligned}$$

The purpose of the work is to analysis the mathematical behavior of the solution $s(t), i(t), r(t)$ for the different values of α . By taking the inverse Laplace transform on both sides of the equation (28), we get the values of s_0, i_0, r_0 and used for further process. Putting the values of s_0, i_0, r_0 and A_0 into the equations (29), (30) and (31), get the values of s_1, i_1, r_1 similarly we find the remaining term $s_2, s_3, s_4 \dots, i_2, i_3, i_4 \dots,$ and $r_2, r_3, r_4 \dots,$ in the same manners. Solution can be written as

$$s(t) = s_0 + s_1 + s_2 + s_3 + s_4 \dots \tag{32}$$

$$i(t) = i_0 + i_1 + i_2 + i_3 + i_4 \dots \tag{33}$$

$$r(t) = r_0 + r_1 + r_2 + r_3 + r_4 \dots \tag{34}$$

The values of parameter used in the computation are given in the following Table 1.

Table 1: Parameter values of SIR measles model

Parameter	Values	Description
N	1000	Total Population
n_1	990	Initial population of $s(t)$
n_2	10	Initial population of $i(t)$
n_3	0	Initial population of $r(t)$
γ	1	Per capita recovery rate
β	0.003	Transitivity
μ	0.05	Per capita removal rate

We have computed first three terms of the series by using the LADM for the equations (4), (5) and (6) as

$$\begin{aligned} s_0 &= 990 + \frac{t^{\alpha_1}}{\alpha_1!}, \quad i_0 = 10, \quad r_0 = 0 \\ s_1 &= -19.8 \frac{t^{\alpha_1}}{\alpha_1!} - 0.05 \frac{t^{2\alpha_1}}{2\alpha_1!}, \quad i_1 = 19.2 \frac{t^{\alpha_2}}{\alpha_2!}, \quad r_1 = -10 \frac{t^{\alpha_3}}{\alpha_3!} \\ s_2 &= -57.024 \frac{t^{\alpha_1+\alpha_2}}{(\alpha_1+\alpha_2)!} - 0.0576 \frac{(\alpha_1+\alpha_2)! t^{\alpha_1+\alpha_2}}{\alpha_1! \alpha_2! (2\alpha_1+\alpha_2)!} + 1.538 \frac{t^{2\alpha_1}}{2\alpha_1!} + \\ & 0.004 \frac{t^{3\alpha_1}}{3\alpha_1!}, \\ i_2 &= 36.864 \frac{t^{2\alpha_2}}{2\alpha_2!} + 0.0576 \frac{(\alpha_1+\alpha_2)! t^{\alpha_1+2\alpha_2}}{\alpha_1! \alpha_2! (\alpha_1+2\alpha_2)!} - 1.538 \frac{t^{\alpha_1+\alpha_2}}{(\alpha_1+\alpha_2)!} - \\ & 0.004 \frac{t^{2\alpha_1+\alpha_2}}{(2\alpha_1+\alpha_2)!}, \\ r_2 &= 19.2 \frac{t^{\alpha_2+\alpha_3}}{(\alpha_2+\alpha_3)!} - 0.5 \frac{t^{2\alpha_3}}{2\alpha_3!} \end{aligned}$$

The solution of the fractional model in series form is written as

$$\begin{aligned} s(t) &= 990 - 19.8 \frac{t^{\alpha_1}}{\alpha_1!} - 0.05 \frac{t^{2\alpha_1}}{2\alpha_1!} - 57.024 \frac{t^{\alpha_1+\alpha_2}}{(\alpha_1+\alpha_2)!} - \\ & 0.0576 \frac{(\alpha_1+\alpha_2)! t^{\alpha_1+\alpha_2}}{\alpha_1! \alpha_2! (2\alpha_1+\alpha_2)!} + 1.538 \frac{t^{2\alpha_1}}{2\alpha_1!} + 0.004 \frac{t^{3\alpha_1}}{3\alpha_1!} & (35) \end{aligned}$$

$$\begin{aligned} i(t) &= 10 + 19.2 \frac{t^{\alpha_2}}{\alpha_2!} + 36.864 \frac{t^{2\alpha_2}}{2\alpha_2!} + 0.0576 \frac{(\alpha_1+\alpha_2)! t^{\alpha_1+2\alpha_2}}{\alpha_1! \alpha_2! (\alpha_1+2\alpha_2)!} - \\ & 1.538 \frac{t^{\alpha_1+\alpha_2}}{(\alpha_1+\alpha_2)!} - 0.004 \frac{t^{2\alpha_1+\alpha_2}}{(2\alpha_1+\alpha_2)!} & (36) \end{aligned}$$

$$r(t) = -10 \frac{t^{\alpha_3}}{\alpha_3!} + 19.2 \frac{t^{\alpha_2+\alpha_3}}{(\alpha_2+\alpha_3)!} - 0.5 \frac{t^{2\alpha_3}}{2\alpha_3!} \tag{37}$$

In particular, the solution of the model with fractional derivatives for $\alpha_1 + \alpha_2 + \alpha_3 = 1$ is given by

$$s(t) = 990 - 18.6t - 27.565t^2 - 0.01853t^3 \tag{38}$$

$$i(t) = 10 + 19.2t + 17.6t^2 + 0.01833t^3 \tag{39}$$

$$r(t) = 10t + 9.35t^2 \tag{40}$$

4. Numerical results and discussion

The numerical results of susceptible, infected and recovered population for $\alpha_i = 1, \alpha_j = 0.99, \alpha_k = 0.95$ where $i, j, k = 1, 2, 3$ are established in Tables 2-4 by using LADM. For the reliable investigation, evaluation is made for different values of α . From Figs. 1-3, we observe that fractional order SIR measles model has more degree of freedom as compared to ordinary derivatives. By taking non-integer values of fractional parameter, remarkable responses of the compartments of the proposed model are obtained. Another remarkable point to be considered that we used small interval of time because we have assumed comparatively small initial values. For large interval of time, the initial values to data are taken large so that the population may not be negative. For different values of α solution converges to steady state and gives the better convergence by decreasing the fractional values of α .

4.1. Convergence analysis

The obtained series solution is rapidly convergent and also converges uniformly to the exact solution. We use the classical techniques to verify the convergence of the series (35), (36) and (37) in (Shah et al., 2016). We check the condition of

convergence of the method by using the idea of the following theorem (Abdelrazec and Pelinovsky, 2011; Naghipour and Manafian, 2015).

Theorem 4.1: Let Y be a Banach space and $F: Y \rightarrow Y$ be a contractive nonlinear operator then there exist $y, y' \in Y, \|F(y) - F(y')\| \leq k\|y - y'\|, 0 < k < 1$. Then F has a unique point y such that $Fy = y$, where $y = (s, i, r)$. The series given in (35), (36) and (37) by using ADM technique is given as:

$$y_n = Ty_{n-1}, y_{n-1} = \sum_{j=1}^{n-1} y_j, n = 1, 2, 3, \dots$$

and suppose that $y_0 \in B_r(y)$ where $B_r(y) = \{y' \in Y: \|y - y'\| < r\}$ then we get

- (i) $y_n \in B_r(y)$
- (ii) $\lim_{n \rightarrow \infty} y_n = y$

Proof: For (i) by using mathematical induction for $n = 1$, we obtained

$$\|y_0 - y\| = \|F(y_0) - F(x)\| \leq k\|y_0 - y\|$$

suppose that the statement is true for $m - 1$ then,

$$\|y_0 - y\| \leq k^{m-1}\|y_0 - y\|$$

we get

$$\|y_m - y\| = \|F(y_{m-1}) - F(x)\| \leq k\|y_{m-1} - y\| \leq k^m\|y_0 - y\|$$

$$\|y_m - y\| \leq k^n\|y_0 - y\| \leq k^n r \leq r$$

which implies that $y_n \in B_r(y)$

- (ii) Since $\|y_m - y\| \leq k^n\|y_0 - y\|$ and $\lim_{n \rightarrow \infty} k^n = 0$ therefore, we have the $\lim_{n \rightarrow \infty} \|y_n - y\| = 0 \Rightarrow \lim_{n \rightarrow \infty} y_n = y$

Table 2: Table of $s(t)$ at different values of α

T	$\alpha_1 = 1$ $\alpha_2 = 1$ $\alpha_3 = 1$	$\alpha_1 = 0.99$ $\alpha_2 = 0.99$ $\alpha_3 = 0.99$	$\alpha_1 = 0.95$ $\alpha_2 = 0.95$ $\alpha_3 = 0.95$	$\alpha_1 = 1$ $\alpha_2 = 0.99$ $\alpha_3 = 0.99$	$\alpha_1 = 1$ $\alpha_2 = 0.95$ $\alpha_3 = 0.95$	$\alpha_1 = 0.99$ $\alpha_2 = 0.99$ $\alpha_3 = 0.95$
0	990	990	990	990	990	990
1	943.616	943.541	940.429	943.173	942.105	942.842
2	841.992	843.622	839.457	841.014	840.014	840.872
3	685.015	690.814	690.18	683.358	685.781	684.677
4	472.574	485.419	494.342	471.83	480.415	474.57
5	204.559	227.625	253.126	204.126	224.574	210.741

Table 3: Table of $i(t)$ at different values of α

T	$\alpha_1 = 1$ $\alpha_2 = 1$ $\alpha_3 = 1$	$\alpha_1 = 0.99$ $\alpha_2 = 0.99$ $\alpha_3 = 0.99$	$\alpha_1 = 0.95$ $\alpha_2 = 0.95$ $\alpha_3 = 0.95$	$\alpha_1 = 1$ $\alpha_2 = 0.99$ $\alpha_3 = 0.99$	$\alpha_1 = 1$ $\alpha_2 = 0.95$ $\alpha_3 = 0.95$	$\alpha_1 = 0.99$ $\alpha_2 = 0.99$ $\alpha_3 = 0.95$
0	10	10	10	10	10	10
1	46.8583	47.267	48.9213	47.2682	48.9584	49.267
2	119.107	119.323	120.059	119.282	120.091	127.213
3	226.855	225.896	221.803	225.725	221.739	243.505
4	370.213	366.84	353.14	366.409	352.875	397.965
5	549.291	542.076	513.378	541.22	512.797	590.492

Table 4: Table of $r(t)$ at different values of α

T	$\alpha_1 = 1$ $\alpha_2 = 1$ $\alpha_3 = 1$	$\alpha_1 = 0.99$ $\alpha_2 = 0.99$ $\alpha_3 = 0.99$	$\alpha_1 = 0.95$ $\alpha_2 = 0.95$ $\alpha_3 = 0.95$	$\alpha_1 = 1$ $\alpha_2 = 0.99$ $\alpha_3 = 0.99$	$\alpha_1 = 1$ $\alpha_2 = 0.95$ $\alpha_3 = 0.95$	$\alpha_1 = 0.99$ $\alpha_2 = 0.99$ $\alpha_3 = 0.95$
0	0	0	0	0	0	0
1	19.35	19.565	20.438	19.565	20.438	20.070
2	57.4	57.514	57.906	57.514	57.906	57.599
3	114.15	113.645	11.495	113.645	111.495	112.206
4	189.6	187.823	180.622	187.823	180.622	183.556
5	283.75	279.952	264.876	279.952	264.876	271.401

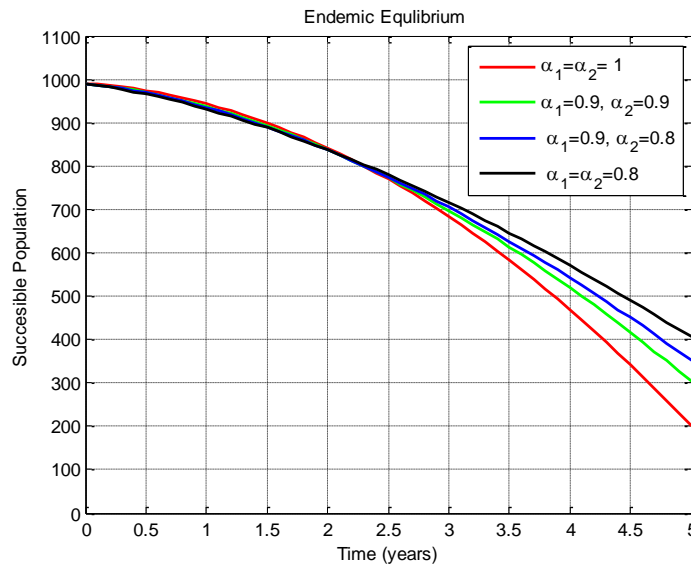


Fig. 1: Numerical solution for susceptible $s(t)$ population in a time t (year) at $\alpha_i = 1, \alpha_j = 0.99, \alpha_k = 0.95$ where $i, j, k = 1, 2, 3$

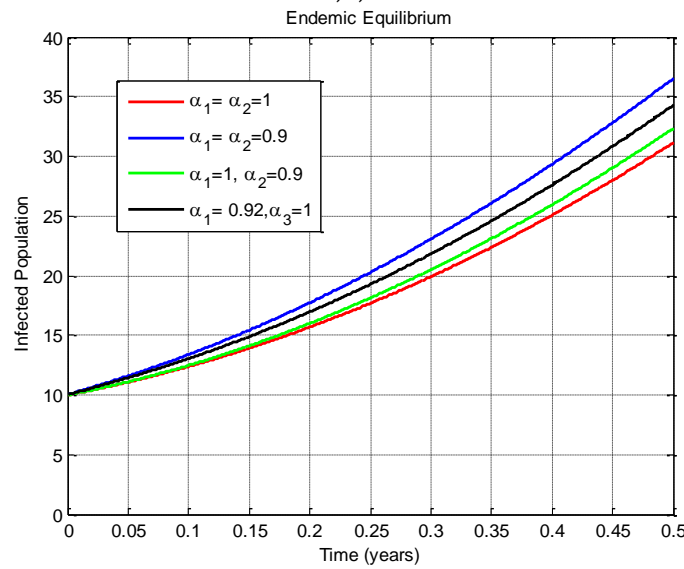


Fig. 2: Numerical solution for infected $i(t)$ population in a time t (year) at $\alpha_i = 1, \alpha_j = 0.99, \alpha_k = 0.95$ where $i, j, k = 1, 2, 3$

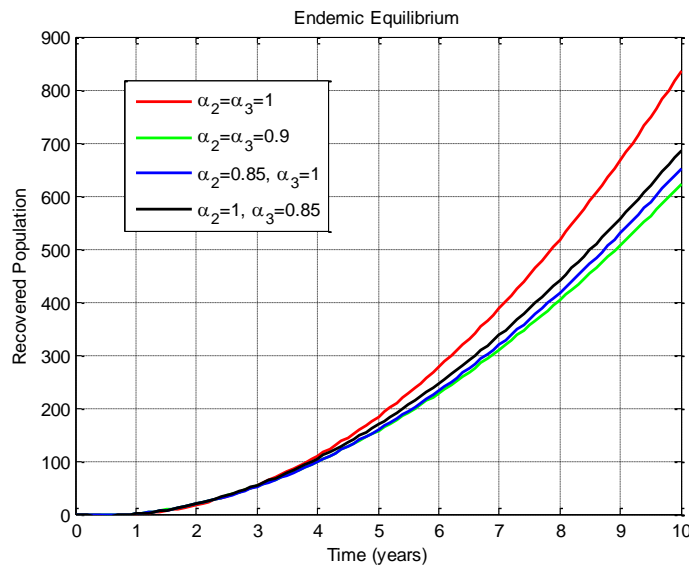


Fig. 3: Numerical solution for recovered $r(t)$ population in a time t (year) at $\alpha_i = 1, \alpha_j = 0.99, \alpha_k = 0.95$ where $i, j, k = 1, 2, 3$

5. Conclusion

In this paper, we developed a scheme for analytical solution of epidemic fractional SIR model of measles by using Laplace Adomian decomposition method. The well-known epidemic model namely Susceptible-Infected-Recovered (SIR) is considered with and without demographic effects. The model represents population dynamics during the disease as a set of non-linear coupled ordinary differential equations. There is no exact solution available in the literature for this model up to the best of author's knowledge. It is observed that the infection rate and reproductive numbers play a key role for an epidemic to occur and the epidemic can be controlled by vaccination. It is also observed that to eliminate the disease, it is not necessary to vaccinate whole of the population. The efficiency and accuracy of the proposed scheme is provided by performing convergence analysis. The effect of fractional parameter on our obtained solutions is presented through Tables and graphs. It is worthy to observe that fractional derivatives show significant changes and memory effects as compared to ordinary derivatives.

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